Report of Meeting

Aczél100
Hotel Aurum, Hajdúszoboszló (Hungary), February 2–7, 2025
Dedicated to the 100th anniversary of the birth of Professor János Aczél

The Aczél100 conference was held in Hajdúszoboszló, Hungary, from February 2 till February 7, 2025, at Hotel Aurum. The meeting was organized by the Institute of Mathematics of the University of Debrecen.

The conference was sponsored by the Mecenatúra project "Aczél 100".[1]

The Organizing Committee consisted of Professors Zsolt Páles as Chairman, Zoltán Boros and Attila Gilányi as Vice-Chairmen, Eszter Gselmann as Project Manager, Gergő Nagy as Scientific Secretary, Tibor Kiss as Technical Secretary, Mihály Bessenyei, Richárd Grünwald, Mehak Iqbal, and Paweł Solarz as Web Administrator.

The Scientific Committee consisted of Professors Gian Luigi Forti, Roman Ger, Zsolt Páles, Jürg Rätz, Ludwig Reich, Maciej Sablik, Jens Schwaiger and László Székelyhidi.

The 52 participants came from Austria, Canada, Hungary, India, Italy, Japan, Poland, Romania and Spain.

The meeting was opened by Zsolt Páles who welcomed the participants on behalf of the Scientific Committee, and highlighted that the conference is dedicated to Professor János Aczél on the occasion of the 100th anniversary of his birth on 26th December 2024.

The scientific program was highlighted by five plenary talks. In one of the presentations, entitled "The Life of János Aczél", Zsolt Páles gave an overview of the scientific career of János Aczél. The regular talks focused on the following topics: functional equations in a single and in several variables; utility theory; Hyers-Ulam stability; convexity; mean values; functional equations on algebraic structures; fixed point theory; iteration theory; probability theory; functional analysis; matrix theory; mathemathical physics; applications of functional equations in topics related to economics, game theory and sociology. Several sessions were devoted to problems and remarks.

On Tuesday evening, the banquet was organized in the aula of the University of Debrecen. It was attended by several leaders of the Institute of Mathematics. The banquet was inaugurated by

[1]Project no. MEC_SZ 148984 has been implemented with the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the MEC_SZ_24 funding scheme.

National Research, Development and Innovation Office Hungary
PROJECT
FINANCED FROM
THE NRDI FUND

Mecenatúra

a concert given by some students of Éva Bátori, accompanied by Dávid Kozma. Gian Luigi Forti gave the banquet address during which he, among others, thanked the organizing committee for their hospitality.

On Wednesday, a flute concert was given by Professor Harald Fripertinger.

In the last section of the conference, some participants recalled personal aspects of the life of János Aczél, and Maciej Sablik closed the event.

Abstracts of the talks follow in alphabetical order of the authors. These are followed by the problems and remarks, in chronological order, and finally, the list of participants.

1. ABSTRACTS

Plenary talks. CHUDZIAK, JACEK: *Biseparable representations of certainty equivalents.* (Joint work with Michał Lewandowski.)

Assume that $(x, y; p)$ is a binary monetary lottery that pays x with probability p, and y with probability $1 - p$. Let $F(x, y; p)$ denote the certainty equivalent of $(x, y; p)$, i.e. a certain amount, the receipt of which is as good for the decision maker as playing the lottery. We investigate individual preferences that lead to the following representation of the certainty equivalent

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)), \tag{1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing utility function and $w : [0, 1] \to [0, 1]$ is a probability weight function.

We consider various axioms leading to (1). In particular, we characterize (1) in the domain of all binary lotteries as well as in the subset of simple lotteries, where one of the two payouts of a lottery is fixed. Furthermore, in the domain of all binary lotteries we separately characterize the case where $F(x, y; p)$ depends on the rank of the payouts (rank dependency) and the case where it does not. Finally we present an analysis of the relationships between the proposed axioms and several versions of the bisymmetry-like axiom.

GER, ROMAN: *Erdős number.*

Due to Professor **János Aczél** and the paper [1], my Erdős number is 2. Like numerous members of this meeting I am very proud of that fact. I do believe that it justifies the idea of evoking some details of [1] showing that the problem of determining all utility measures over binary gambles that are both separable and additive leads to the functional equation

$$f(v) = f(vw) + f[vQ(w)]\,, \quad v, vQ(w) \in [0, k)\,, w \in [0, 1]\,.$$

The following conditions seem to be more or less natural to that problem: f is strictly increasing, Q is strictly decreasing, both map their domains onto intervals (f onto a $[0, K), Q$ onto $[0, 1]$), thus both are continuous, $k > 1, f(0) = 0, f(1) = 1, Q(1) = 0, Q(0) = 1$. We determine, however, the general solution without any of these conditions (except $f : [0, k) \longrightarrow \mathbb{R}_+ := [0, \infty)$, $Q : [0, 1] \longrightarrow \mathbb{R}_+$, both into). If we exclude two trivial solutions, then we get as the general solution $f(v) = \alpha v^{\beta}$ ($\beta > 0, \alpha > 0$; $\alpha = 1$ for $f(1) = 1$), which satisfies all the above conditions.

The paper concludes with a remark on the case where the equation is satisfied only almost everywhere.

In the second part of my talk I take the liberty of telling the audience some of my less formal contacts with János (like in [2]).

References

[1] J. Aczél, R. Ger, A. Járai, *Solution of a functional equation arising from utility that is both separable and additive*, Proceedings of the American Mathematical Society 127 (1998), 2923–2929.
[2] J. Aczél, *A remark in a private correspondence*.

NG, CHE TAT: *Measuring movement of incomes.*

Professor Aczél has shown us the usefulness of functional equations in the theory of measurements. The axioms involved in the characterization and derivation of a measure may be qualitative or quantitative in nature. Functional inequalities also play their role. I shall bring up some of them which are found in the literature for discussion.

References

[1] C.P. Niculescu, L.-E. Persson, *Convex functions and their applications, A contemporary approach*, Springer, Berlin, Heidelberg, New York, 2004.
[2] F.A. Cowell, E. Flachaire, *Measuring mobility*, Quantitative Economics 9 (2) (2018), 865–901.
[3] G.S. Fields, E.A. Ok, *Measuring movement of incomes*, Economica 66 (1999), 455–471.
[4] P.C. Fishburn, *Utility Theory for Decision Making*, Wiley, New York, 1970.
[5] T. Mitra, E.A. Ok, *The Measurement of Income Mobility: A Partial Ordering Approach*, Economic Theory 12 (1998), 77–102.

PÁLES, ZSOLT: *Contributions of János Aczél to the theory of means.*

One of the most significant results of the theory of quasiarithmetic means, their characterization theorem, was discovered independently by de Finetti [4], Kolmogorov [5] and Nagumo [6] in 1930-31. Each of these characterizations involved quasiarithmetic means with a non-fixed number of variables. János Aczél's main result was published [1] in 1947, where he introduced the notion of bisymmetry which turned out to be the central concept for the characterization of quasiarithmetic means with a fixed number of variables. This paper was the starting point of a new research direction in the theory of means and had many applications in aggregation and decision theory.

One of the important generalizations of quasiarithmetic means was introduced by Bajraktarević [3] in 1963. Motivated by the properties of the so-called Rényi entropies and their applications in information theory, the characterization of homogeneity of these means became a fundamental question which was answered by Aczél and Daróczy [2] in 1963.

In the talk, we aim to describe the above mentioned results in details and sketch some of their applications.

References

[1] J. Aczél, *The notion of mean values*, Norske Vid. Selsk. Forh., Trondhjem 19 (23) (1947), 83–86.
[2] J. Aczél and Z. Daróczy, *Über verallgemeinerte quasilineare Mittelwerte, die mit Gewichtsfunktionen gebildet sind*, Publ. Math. Debrecen 10 (1963), 171–190.
[3] M. Bajraktarević, *Sur une généralisation des moyennes quasilinéaires*, Publ. Inst. Math. (Beograd) (N.S.) 3 (17) (1963), 69–76.

[4] B. de Finetti, *Sul concetto di media*, Giornale dell' Instituto, Italiano degli Attuarii 2 (1931), 369–396.
[5] A.N. Kolmogorov, *Sur la notion de la moyenne*, Rend. Accad. dei Lincei 12 (6) (1930), 388–391.
[6] M. Nagumo, *Über eine Klasse der Mittelwerte*, Japanese J. Math. 7 (1930), 71–79.

Regular talks. ACU, ANA-MARIA: *Types of convexity related to a C_0-semigroup.* (Joint work with Georgian Chivu, Ioan Raşa.)

Let K be a convex compact subset of $\mathbb{R}^p$, $p \geq 1$, having nonempty interior. Starting with a suitable positive linear projection T defined on $C(K)$, Altomare and Raşa defined in [1] weakly T-convex functions. Using T, a C_0-semigroup of operators on $C(K)$ was constructed and the generalized A-subharmonic functions were defined, where A is the infinitesimal generator of the semigroup. It was proved that if a function is weakly T-convex, then it is generalized A-subharmonic. The authors of [1] conjectured that the converse is also true, but as far as we know this is still an open problem. We present some results related to the conjecture. Namely, starting with the conjecture, we prove that a suitable stronger hypothesis entails a stronger conclusion. This study extends certain results from [1].

REFERENCES

[1] A.M. Acu, I. Raşa, *Generalized Subharmonic and Weakly Convex Functions*, In: A.M. Candela, M. Cappelletti Montano, E. Mangino (eds.) Recent Advances in Mathematical Analysis, Trends in Mathematics, Birkhäuser, Cham, 2023.
[2] F. Altomare, I. Raşa, *Feller semigroups, Bernstein type operators and generalized convexity associated with positive projections*, New Developments in Approximation Theory (Dortmund, 1998), Internat. Ser. Numer. Math. 132, Birkhäuser, Basel, 1999, 9–32.

ALMIRA, JOSÉ MARÍA: *Solution to a conjecture of Laird and McCann.*

Let X_d denote indistinctly either the space $\mathcal{D}(\mathbb{R}^d)'$ of Schwartz complex valued distributions defined on $\mathbb{R}^d$ or the space $C(\mathbb{R}^d)$ of continuous complex valued functions defined on $\mathbb{R}^d$. For $f \in \mathcal{D}(\mathbb{R}^d)'$ we introduce the operators

$$\tau_h(f)\{\phi\} = f\{\tau_{-h}(\phi)\}, \text{ where } h \in \mathbb{R}^d \text{ and } \phi \in \mathcal{D}(\mathbb{R}^d) \text{ is any test function,}$$

and $\tau_{-h}(\phi)(x) = \phi(x-h)$. We also consider the operators

$$O_P(f)\{\phi\} = \frac{1}{|\det(P)|} f\{O_{P^{-1}}(\phi)\},$$

where $P \in \mathbf{GL}_d(\mathbb{E})$ is any invertible matrix, $\phi \in \mathcal{D}(\mathbb{R}^d)$ is any test function, and

$$O_{P^{-1}}(\phi)(x) = \phi(P^{-1}x) \quad \text{ for all } x \in \mathbb{R}^d$$

In the talk we will demonstrate the following result, which generalizes a theorem by Loewner [3, 4] to distributions and whose proof solves a conjecture posed by Laird and McCann [2] in 1984.

Theorem Let $d \geq 2$ be a natural number, let $f \in X_d$ and assume that, for a certain finite dimensional space $V \subseteq X_d$ we have that:

- $f \in V$
- V is translation invariant (i.e., $\tau_h(V) \subseteq V$ for all $h \in \mathbb{R}^d$).

- V is invariant under orthogonal transformations of $\mathbb{R}^d$ (i.e., $O_P(V) \subseteq V$, for all $P \in \mathbf{O}(d)$).

Then f is, in distributional sense, an ordinary polynomial on $\mathbb{R}^d$. In particular, f is equal almost everywhere to an ordinary polynomial and, if f is a continuous ordinary function, then it is an ordinary polynomial.

REFERENCES

[1] J. M. Almira, *On Loewner's characterization of polynomials*, Jaen J. Approx. 8 (2) (2016), 175-–181.

[2] P.G. Laird, R. McCann, *On some characterizations of polynomials*, Amer. Math. Monthly 91 (2) (1984), 114–116.

[3] P.G. Laird, *On characterizations of exponential polynomials*, Pacific J. Math. 80 (1979), 503–507.

[4] C. Loewner, *On some transformation semigroups invariant under Euclidean and non-Euclidean isometries*, J. Math. Mech. 8 (1959), 393–409.

BARON, KAROL: *Strong law of large numbers for iterates of weakly contractive in mean random-valued functions.* (Joint work with Rafał Kapica.)

Assume $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space, (X, ρ) is a complete and separable metric space with the σ-algebra $\mathcal{B}$ of all its Borel subsets and $f : X \times \Omega \to X$ is measurable for $\mathcal{B} \otimes \mathcal{A}$ and such that

$$\int_\Omega \rho\big(f(x,\omega), f(z,\omega)\big)\mathbb{P}(d\omega) \le \beta\big(\rho(x,z)\big) \quad \text{for } x, z \in X$$

with a concave $\beta : [0,\infty) \to [0,\infty)$ satisfying $\sum_{n=1}^{\infty} \beta^n(t) < \infty$ for $t \in (0,\infty)$, and

$$\int_\Omega \rho\big(f(x_0,\omega), x_0\big)\mathbb{P}(d\omega) < \infty \quad \text{for an } x_0 \in X.$$

We consider the sequence of iterates of f defined on $X \times \Omega^{\mathbb{N}}$ by $f^0(x,\omega) = x$ and

$$f^n(x,\omega) = f\big(f^{n-1}(x,\omega), \omega_n\big) \quad \text{for } n \in \mathbb{N},$$

its weak limit π^f and the problem of the almost sure convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ f^k(x,\cdot) = \int_X \psi d\pi^f$$

for $x \in X$ and Lipschitzian $\psi : X \to \mathbb{R}$.

BOROS, ZOLTÁN: *From space perception to utility: the evolution of some functional equations.*

In an axiomatic development of the psychophysical theory of binocular vision [3], some invariance relations led to functional equations of the form

$$F(x + \theta t, y + \sigma t) = H_{\theta,\sigma}(F(x,y), t)\,, \tag{1}$$

where the real valued function F is defined on a given bounded, convex, and open subset of the plane, $\theta, \sigma \in \{-1, 0, 1\}$, and the functions $H_{\theta,\sigma}$ are also unknown. Assuming that F can be represented as the sum of two continuous and strictly monotone real functions and it fulfills a pair of the considered functional equations (1), we proved that F is linear.

Motivated by these results, the present author [4] investigated pairs of equations

$$[(1), (\theta, \sigma) \in \{(1,0), (0,1)\}]$$

for a continuous function $F : S \to \mathbb{R}$, strictly monotone in one of its variables, defined on an arbitrary connected and open set $S \subseteq \mathbb{R}^2$. In this case, every solution has the representation

$$F(x, y) = \varphi(ax + by) \qquad ((x, y) \in S), \tag{2}$$

where a and b are non-zero real numbers and φ is a strictly monotone function defined on an interval.

Péter Tóth [6] has extended the latter result to contimuous functions in several variables (relaxing the monotonicity assumption as well). Applying the representation theorem for continuous, associative and cancellative operations on an interval ([1], [2] and [5]), he could replace the addition on the left-hand side with particular operations, obtaining characterizations of various particular utility functions in terms of systems of functional equations [7].

REFERENCES

[1] J. Aczél, *Sur les opérations définies pour nombres réels*, Bull. Soc. Math. France 76 (1949), 59–64.

[2] J. Aczél, *Lectures on Functional Equations and Their Applications,* Dover Publications, Inc., Mineola, New York, 1966.

[3] J. Aczél, Z. Boros, J. Heller, and C. T. Ng, *Functional Equations in Binocular Space Perception*, J. Math. Psych. 43 (1) (1999), 71–101.

[4] Z. Boros, *Systems of generalized translation equations on a restricted domain*, Aequationes Math. 67 (2004), 106–116.

[5] R. Craigen, Zs. Páles, *The associativity equation revisited*, Aequationes Math. 37 (1989), 306–312.

[6] P. Tóth, *Continuous solutions of a system of composite functional equations*, Aequationes Math. 96 (2022), 1179–1205.

[7] P. Tóth, *Characterizations of utility functions in terms of functional equations* (manuscript).

CHMIELIŃSKI, JACEK: *Norm derivatives in complex spaces.*

In a *real* normed space X the norm derivatives at $x \in X$ in the direction of $y \in X$ are defined as

$$\rho'_{\pm}(x, y) := \lim_{\lambda \to 0^{\pm}} \frac{\|x + \lambda y\|^2 - \|x\|^2}{2\lambda} = \|x\| \lim_{\lambda \to 0^{\pm}} \frac{\|x + \lambda y\| - \|x\|}{\lambda}.$$

Now, if X is a *complex* normed space, then for each $\theta \in [0, 2\pi)$ we define

$$\rho'_{\theta}(x, y) = \lim_{r \to 0^{+}} \frac{\|x + re^{i\theta} y\|^2 - \|x\|^2}{2r} = \|x\| \lim_{r \to 0^{+}} \frac{\|x + re^{i\theta} y\| - \|x\|}{r}.$$

In particular, for $\theta \in \{0, \pi\}$, we have $\rho'_0 = \rho'_+$ and $\rho'_\pi = -\rho'_-$. Thus the notion of complex norm derivatives ρ'_θ extends that of $\rho'_\pm$. We will speak on the properties and applications of the functionals ρ'_θ.

FAZEKAS, BORBÁLA: *On the characterization of turbulent magnetohydrodynamic mean flows.* (Joint work with József Kolumbán.)

We consider the ideal magnetohydrodynamic system

$$\begin{aligned} \partial_t u + \operatorname{div}(uu^T - BB^T) + \nabla p &= 0 \\ \partial_t B + \operatorname{div}(Bu^T - uB^T) &= 0 \\ \operatorname{div} u = \operatorname{div} B &= 0, \end{aligned}$$

where $u, B\colon \mathbb{R}^3 \times (0,T) \to \mathbb{R}^3$, $p\colon \mathbb{R}^3 \times (0,T) \to \mathbb{R}$.

We aim at infinitely many $\mathbb{L}^\infty$ solutions of this system via the so-called Tartar framework. For this purpose the Λ-convex hull of the set

$$K_{r,s} = \{(u,S,B,E) \in \mathbb{R}^{15} \mid S = uu^T - BB^T + pI,\ E = B \times u,$$
$$\|u+B\| = r,\ \|u-B\| = s,\ p \in \mathbb{R},\ |p| \le rs\}$$

with $r, s \in \mathbb{R}$ fixed, is of high importance, where the cone Λ consists of elements $(u,S,B,E) \in \mathbb{R}^{15}$ such that there exist $\xi_x \in \mathbb{R}^3 \setminus \{0\},\ \xi_t \in \mathbb{R}$ with $\xi_t u + S \cdot \xi_x = 0$, $\xi_t B + \xi_x \times E = 0$, $\xi_x \cdot u = 0$, $\xi_x \cdot B = 0$.

We determine the first laminate of $K_{r,s}$ and give an enclosing set for the Λ-convex hull of $K_{r,s}$.

REFERENCES

[1] C. De Lellis, L. Székelyhidi Jr., *Weak stability and closure in turbulence*, (2021), arXiv:2108.01597.

[2] L. Hitruhin, S. Lindberg, *Relaxation of the kinematic Dynamo Equations*, (2023), arXiv:2301.06843.

[3] L. Tartar, *The compensated compactness method applied to systems of conservation laws*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 111, Reidel, Dordrecht (1983), 263–285.

FECHNER, WŁODZIMIERZ: *Two characterizations of quasiconvexity.*

Let X be a real linear space, $D \subset X$ a nonempty convex set, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. A function $f\colon D \to \overline{\mathbb{R}}$ is called quasiconvex if

$$\forall_{x,y\in D} \forall_{t\in[0,1]} f(tx + (1-t)y) \le \max\{f(x), f(y)\}.$$

This talk focuses on characterizing quasiconvexity under assumptions of radial lower semicontinuity or upper semicontinuity. Drawing inspiration from the work of Páles [3], Daróczy and Páles [1] on convexity characterizations (see also Leonetti [2]), we establish, among others, the following result:

Assume that $f\colon D \to \overline{\mathbb{R}}$ is a radially lower semicontinuous function. If

$$\forall_{x,y\in D} \exists_{z\in]x,y[}\, f(z) \le \max\{f(x), f(y)\},$$

then f is quasiconvex on D.

We will show some applications of our findings, including an extension of Sion's minimax theorem [4] and characterizations of quasiconvex risk measures.

REFERENCES

[1] Z. Daróczy, Zs. Páles, *A characterization of nonconvexity and its applications in the theory of quasi-arithmetic means*, Inequalities and applications, Internat. Ser. Numer. Math. 157, Birkhäuser, Basel, (2009), 251–260.

[2] P. Leonetti, *A characterization of convex functions*, arXiv:1709.08611 (2017).

[3] Zs. Páles, *Nonconvex functions and separation by power means*, Math. Inequal. Appl. 3 (2) (2000), 169–176.

[4] M. Sion, *On general minimax theorems*, Pacific J. Math. 8 (1958), 171–176.

FORTI, GIAN LUIGI: *C^2 solutions of an alternative quadratic functional equation.*

The aim of this presentation is the investigation of the following alternative quadratic functional equation:

$$\begin{aligned} &f(x+y)+f(x-y)-2f(x)-2f(y)\neq 0 \quad \text{implies} \\ &\qquad\qquad g(x+y)+g(x-y)-2g(x)-2g(y)=0, \end{aligned}$$

where $f,g:\mathbb{R}\to\mathbb{R}$ are $C^2(\mathbb{R})$ functions. The assumed regularity condition allows a useful integral representation of the quadratic difference of the two functions f and g. Indeed we have

$$\begin{aligned} f(x) &= \int_0^x \tau(t)dt + \frac{C_1}{2}x^2 + C_2x + C_3, \quad C_3 = f(0), \\ g(x) &= \int_0^x \sigma(t)dt + \frac{D_1}{2}x^2 + D_2x + D_3, \quad D_3 = g(0), \end{aligned} \tag{1}$$

where τ and σ are C^1 functions on $\mathbb{R}$, with $\tau(0)=\tau'(0)=0=\sigma(0)=\sigma'(0)$.

Under these conditions we obtain the following result:

Theorem. *The alternative quadratic equation has only trivial solutions, that is either $f(x)=\alpha x^2$ for $x\in\mathbb{R}$, or $g(x)=\beta x^2$ for $x\in\mathbb{R}$, for some real numbers α and β.*

Some open problems are then presented.

REFERENCES

[1] J. Dhombres, *Relations de dépendance entre les équations fonctionnelles de Cauchy*, Aequationes Math. 35 (1988), 186–212.

[2] G.L. Forti, L. Paganoni, *On an alternative Cauchy equation in two unknown functions. Some classes of solutions*, Aequationes Math. 42 (1991), 271–295.

[3] G. L. Forti, *A note on an alternative quadratic equation*, Annales Univ. Budapest., Sect. Comp. 40 (2013), 223–232.

[4] R. Ger, *On alienation of two functional equations of quadratic type*, Aequationes Math. 95 (2021), 1169–1180.

[5] R. Ger, M. Sablik, *Alien functional equations: a selective survey of results*, in J. Brzdęk, K. Ciepliński and Th.M. Rassias (eds.), Developments in Functional Equations and Related Topics, Springer Optimization and Its Applications 124, Springer, 2017, 107–147.

FRIPERTINGER, HARALD: *On n-associative formal power series over rings.* (Joint work with Susan F. El-Deken.)

In [1] n-associative formal power series over $\mathbb{C}$ were studied. This was a generalization of associativity, which is the special case of 2-associativity and was investigated in [2]. These results on 2-associativity were generalized by F. Halter-Koch in [3] for formal power series in two variables over commutative rings. The starting point for all these investigations was Hazewinkel's book [4] in which formal group laws were studied over $\mathbb{Q}$-algebras. Now we present some results on n-associative formal power series over a commutative ring R with 1 following Halter-Koch's ideas.

A formal power series $F(x_1,\dots,x_n)\in R[\![x_1,\dots,x_n]\!]$ in n variables, $n\geq 3$, of order at least 1 is called n-associative, if the following $\binom{n}{2}$ equations

$$F(F(x_1,\dots,x_n),x_{n+1},\dots,x_{2n-1})=\dots=F(x_1,\dots,x_{n-1},F(x_n,x_{n+1},\dots,x_{2n-1}))$$

hold true. The sequence $\varphi_i(x)=F(0^{i-1},x,0^{n-i})$, $i\in\{1,\dots,n\}$, where x stands in the ith position, plays an important role in the study of n-associative series.

This paper appeared in Rendiconti del Circolo Matematico di Palermo Series 2, Vol. 74, Nr 24, 2025.

References

[1] H. Fripertinger, *On n-associative formal power series*, Aequationes mathematicae, 90 (2) (2016), 449–467.

[2] H. Fripertinger, L. Reich, J. Schwaiger, J. Tomaschek, *Associative formal power series in two indeterminates*, Semigroup Forum, 88 (3) (2014), 529–540.

[3] F. Halter-Koch, *Associative power series*, Aequationes mathematicae, 89 (3) (2015), 765–769.

[4] M. Hazewinkel, *Formal groups and applications,* Academic Press, New York, San Francisco, London, 1978.

GILÁNYI, ATTILA: *On a functional equation arising from investigations of utility.*

This talk focuses on the study of the functional equation

$$F_1(t)-F_1(t+s)=F_2[F_3(t)+F_4(s)], \tag{1}$$

which appears in several investigations related to utility (e.g., in [4], [5] and [7]). We also consider some further interesting equations arising during the solution of (1) (cf., [1], [2] and [6]), as well as related problems, particularly the "Program" of studying extensions established by János Aczél ([3]).

References

[1] J. Aczél, *Extension of a generalized Pexider equation*, Proc. Amer. Math. Soc. 133, (2005), 3227–3233.

[2] J. Aczél, *Utility of extension of functional equations - when possible*, J. Math. Psychology 49 (2005), 445–449.

[3] J. Aczél, *5. Remark, Report of meeting*, Aequationes Math. 69 (2005), 183.

[4] J. Aczél, Gy. Maksa, C.T. Ng, Zs. Páles, *A functional equation arising from ranked additive and separable utility*, Proc. Amer. Math. Soc. 129 (2001), 989–998.

[5] A. Gilányi, C.T. Ng, J. Aczél, *On a functional equation arising from comparison of utility representations*, J. Math. Anal. Appl. 304 (2005), 572–583.

[6] A. Lundberg, *On the functional equation $f(\lambda(x)+g(y))=\mu(x)+h(x+y)$*, Aequationes Math. 16 (1977), 21–30.

[7] C.T. Ng, R. D. Luce, J. Aczél, *Functional characterization of basic properties of utility representations*, Monatsh. Math. 135 (2002), 305–319.

GŁAZOWSKA, DOROTA: *Invariance problems for generalized classical weighted means.*

Let $I\subset\mathbb{R}$ be an interval and let $f\colon I\to\mathbb{R}$ and $\mathrm{id}|_I-f$ be strictly increasing functions. Then the bivariable function $A_f\colon I^2\to\mathbb{R}$ given by

$$A_f\,(x,y)=f\,(x)+y-f\,(y)\,,$$

is a strict mean in the interval I. The mean A_f generalizes classical weighted arithmetic means.

Now let $I \subset (0, +\infty)$ be an interval. Let $g\colon I \to (0, +\infty)$ and $h\colon I \to (0, +\infty)$ be strictly increasing functions such that the functions $\frac{\mathrm{id}|_I}{g}$ and $\mathrm{id}|_I - h$ are also increasing. Then the functions $G_g\colon I^2 \to \mathbb{R}$ and $H_h\colon I^2 \to \mathbb{R}$ given, respectively, by

$$G_g(x, y) = \frac{g(x)}{g(y)} y, \quad H_h(x, y) = \frac{xy}{x - h(x) + h(y)},$$

are strict means in the interval I. These means are natural generalizations, respectively, of the classical weighted geometric mean and the classical weighted harmonic mean.

During the talk, the solutions of some invariance problems involving these means will be presented. Among others, we consider the invariance of a generalized weighted arithmetic mean A_f with respect to a generalized weighted arithmetic mean-type mapping (A_g, A_h), i.e. we solve the equation

$$A_f \circ (A_g, A_h) = A_f,$$

under some additional regularity assumptions on the unknown functions f, g, h. The form of the generalized weighted geometric mean allows us to apply these results, to obtain the relevant solutions of the following invariance equation

$$G_f \circ (G_g, G_h) = G_f,$$

involving only generalized weighted geometric means.

Finally, under general conditions ensuring that A_f, G_g and H_h are means, assuming that f and h are twice continuously differentiable, g is three times continuously differentiable in $I \subset (0, \infty)$, we examine when the equality

$$G_g \circ (H_h, A_f) = G_g,$$

holds, i.e. when the mean G_g is invariant with respect to the mean-type mapping (H_h, A_f).

This is a report on the research made jointly with Janusz Matkowski.

GOPALAKRISHNA, CHAITANYA: *Iterative roots of multifunctions.*

In this talk we discuss some new results on the nonexistence of iterative roots of multifunctions on arbitrary sets. First, we introduce a fundamental class of multifunctions, called pullback multifunctions, and show that the iterative root problem for such multifunctions can be reduced to the corresponding problem for single-valued maps. Then, focusing on multifunctions that are not pullbacks, we prove that typically if the graph of a multifunction has a special point with a relatively large number of paths leading to it then such a multifunction does not admit any iterative root. Notably, we impose no constraints on the number of set-valued points of the map under consideration. We also illustrate our results with some examples, particularly showing the nonexistence of iterative roots of some specified orders for certain complex polynomials and their pullbacks. The talk is based on our recent work [1].

REFERENCES

[1] B.V.R. Bhat, C. Gopalakrishna, *Iterative roots of multifunctions*, Fundamenta Mathematicae 265 (2024), 141–163.

HORVÁTH, LÁSZLÓ: *Majorization type inequalities using signed measures.*

In this talk we derive majorization type integral inequalities using measure spaces with signed measures. We obtain necessary and sufficient conditions for the studied integral inequalities to be satisfied. To apply our results, we first generalize the Hardy-Littlewood-Pólya and Fuchs inequalities. Then we deal with the nonnegativity of some integrals with nonnegative convex functions. As a consequence, the known characterization of Steffensen-Popoviciu measures on compact intervals is extended to arbitrary intervals. Finally, we give necessary and sufficient conditions for the satisfaction of the integral Jensen inequality and the integral Lah-Ribarič inequality for signed measures. All the considered problems are also studied for special classes of convex functions.

JABŁOŃSKI, WOJCIECH: *On substitution in rings of formal power series.*

It is known that the set $\boldsymbol{R}[\![x_1,\dots,x_n]\!]$ of all formal power series of n variables $x_1,\dots,x_n$ over a commutative ring $\boldsymbol{R}$ is a ring with the usual operations of addition and multiplication of formal power series. Moreover, with no additional assumptions one can define the operation of substitution in the ideal $\mathfrak{m}=(x_1,\dots,x_n)$. Following results from [3, 1, 2] we discuss the existence of a substitution in the ring of formal power series of several variables over a ring with a metric. We show that the sufficient and necessary condition for the existence of a substitution which appears in the mentioned papers is natural and almost obvious also in the ring of formal power series in several variables.

REFERENCES

[1] M. Borkowski, P. Maćkowiak, *Further remarks on formal power series*, Comment. Math. Univ. Carolin. 53 (4) (2012), 549–555.

[2] D. Bugajewski, A. Galimberti, P. Maćkowiak, *On composition and Right Distributive Law for formal power series of multiple variables*, 15 pp. arxiv.org/abs/2211.06879

[3] X.-X. Gan, N. Knox, *On composition of formal power series*, Int. J. Math. Math. Sci. 30 (12) (2002), 761–770.

KISS, GERGELY: *Two theorems of Aczél and continuity.* (Joint work with Patricia Szokol and Pál Burai.)

In this talk, I will first present two renowned theorems of Aczél concerning the characterization of quasi-arithmetic means and quasi-sums. Then, I will introduce and discuss our simplified characterization of quasi-arithmetic means without assuming continuity, and I will formulate several potentially open questions posed in our first joint paper [4]. Additionally, I will discuss some dichotomy results arising from the relaxation of symmetry and reflexivity, which provide more nuanced descriptions of quasi-arithmetic means. I will also present a construction that, among other significant consequences, demonstrates that continuity in Aczél's theorem on quasi-sums might not be eliminated. Finally, I will outline some open research directions.

REFERENCES

[1] J. Aczél, *On mean values*, Bull. Amer. Math. Soc. 54 (1948), 392–400.

[2] J. Aczél, *Lectures on Functional Equations and their Applications*, Academic Press, New York and London, 1966.

[3] J. Aczél, J. Dhombres, *Functional equations in several variables*, Encyclopedia of Mathematics and its Applications 31, Cambridge University Press, Cambridge, 1989.

[4] P. Burai, G. Kiss, P. Szokol, *Characterization of quasi-arithmetic means without regularity condition*, Acta Math. Hung. 165 (2021), 474–485.
[5] P. Burai, G. Kiss, P. Szokol, *A dichotomy result for strictly increasing bisymmetric maps*, Journal of Mathematical Analysis and Applications (2023), 127269.

LACZKOVICH, MIKLÓS: *Operations on $\mathbb{R}^n$ isomorphic to $(\mathbb{R}^n, +)$, and the resultant of forces.*

Let $\oplus$ be a binary operation on $\mathbb{R}^n$. When will $(\mathbb{R}^n, \oplus)$ be isomorphic to the Abelian group $(\mathbb{R}^n, +)$? An obvious necessary condition is that $(\mathbb{R}^n, \oplus)$ is a commutative semigroup with unit.

We show that, assuming the condition above and $n \geq 3$, the following is sufficient: $\oplus$ commutes with the elements of SO_n (the family of orthogonal transformations of $\mathbb{R}^n$ with determinant 1); that is, $(Aa) \oplus (Ab) = A(a \oplus b)$ for every $a, b \in \mathbb{R}^n$ and $A \in SO_n$.

The result is closely connected to the characterization of vector addition given by D'Alembert, Darboux and others.

LEWANDOWSKI, MICHAŁ: *Biseparable representations of certainty equivalents – an introduction.* (Joint work with Jacek Chudziak.)

We consider the following biseparable representation of the certainty equivalent:

$$F(x, y; p) = u^{-1}(w(p)u(x) + (1 - w(p))u(y)), \tag{1}$$

where $(x, y; p)$ is a binary monetary prospect, u is a utility function, and w is a probability weight function. We characterize this model in the domain of all binary prospects, as well as in a subset of simple prospects, i.e. prospects in which one of the two payoffs is fixed. For both domains, we consider the case of a general probability weight function w, corresponding to the certainty equivalent in the rank-dependent utility model, as well as several others, e.g.

a. a rank-independent model, in which w is self-conjugate,
b. an anticipated utility model, in which w satisfies the condition $w(0.5) = 0.5$,
c. an expected utility model, in which w is the identity function.

This talk will serve as an introduction and motivation for the plenary talk by Jacek Chudziak, who will focus on our theoretical results concerning the characterization of these different models. I will show why the models we characterize are so important and outline how our results and approach can help with issues related to model identification and distinguishing it from the process of model validation in the context of experimental methods for preference elicitation. I will show that the preferences generating certainty equivalents of binary prospects in the form (1) are very general. In particular, they are consistent with the phenomenon of preference reversal. Even assuming that preferences for prospects are represented by certainty equivalents of prospects, they encompass many popular nonexpected preference models. By focusing on model (1), we investigate the "common denominator" of all these models. I will also outline the problem of cross-domain model extensions.

REFERENCES

[1] P. Wakker, *Prospect theory: For risk and ambiguity*, Cambridge University Press, 2010.

LOSONCZI, LÁSZLÓ: *Products and inverses of multidiagonal matrices with equally spaced diagonals.*

Let n, k be fixed natural numbers with $1 \leq k \leq n$ and let $A_{n+1,k,2k,\ldots,sk}$ denote a complex $(n+1) \times (n+1)$ multidiagonal matrix having $s = [n/k]$ sub- and superdiagonals at distances $k, 2k, \ldots, sk$ from the main diagonal. We prove that the set $\mathcal{MD}_{n,k}$ of all such multidiagonal matrices is closed under multiplication and powers with positive exponents. Moreover the subset of $\mathcal{MD}_{n,k}$ consisting of all nonsingular matrices is closed under taking inverses and powers with negative exponents. In particular, we obtain that the inverse of a nonsingular matrix $A_{n+1,k}$ (called k-tridiagonal) is in $\mathcal{MD}_{n,k}$, moreover if $n+1 \leq 2k$ then $A_{n+1,k}^{-1}$ is also k-tridiagonal. Using this fact, we give an explicit formula for this inverse.

REFERENCES

[1] N. Bebiano, S. Furtado, *A reducing approach for symmetrically sparse banded and anti-banded matrices*, Linear Algebra Appl. 581 (2019), 36–50.

[2] F. Diele, L. Lopez, *The use of the factorization of five-diagonal matrices by tridiagonal Toeplitz matrices*, Appl. Math. Lett. 11 (3) (1998), 61–69.

[3] M. El-Mikkawy, F. Atlan, *A fast and reliable algorithm for evaluating n-th order k-tridiagonal determinants*, Malaysian J. Math. Sci. 3 (2015), 349–365.

[4] C.M. da Fonseca, L. Losonczi, *On the determinant of general pentadiagonal matrices*, Publ. Math. (Debrecen) 97 (3-4) (2020), 507–523.

[5] C.M. da Fonseca, L. Losonczi, *On some pentadiagonal matrices: their determinants and inverses*, Annales Univ. Sci. Budapest., Sect. Comp. 51 (2020), 39–50.

[6] R.B. Marr, G.H. Vineyard, *Five-diagonal Toeplitz determinants and their relation to Chebyshev polynomials*, SIAM J. Matrix Anal. Appl. 9 (4) (1988), 579–586.

[7] J.M. Montaner, M. Alfaro, *On five-diagonal Toeplitz matrices and orthogonal polynomials on the unit circle*, Numer. Algorithms 10 (1995), 137–153.

MATKOWSKI, JANUSZ: *Weak associativity: new examples and open questions.*

In a recent paper [1] it is observed that every weighted quasiarithmetic mean is *weakly associative*, i.e. it satisfies the equality

$$M\left(M\left(x,y\right),x\right) = M\left(x, M\left(y,x\right)\right),$$

and is not a symmetric function. In the present talk we present a new class of weakly associative means and propose some open questions.

REFERENCES

[1] D. Głazowska, J. Matkowski, *Weakly associative functions* (submitted).

MOLNÁR, LAJOS: *Maps on positive cones in operator algebras preserving relative entropies.*

We consider several concepts of (not only numerical valued) relative entropies on positive cones in C^*-algebras. We present recent (not yet published) results showing that surjective transformations between positive cones that preserve any of those quantities necessarily originate from

Jordan *-isomorphisms between the underlying full algebras. In the special case of matrix algebras, we consider the problem of relaxing the condition of surjectivity.

OKAMURA, KAZUKI: *On expectations of power means of random variables.* (Joint work with Yoshiki Otobe.)

In general, it is hard to compute or characterize the expectation of a power mean of independent and identically distributed random variables, due to the fractional power of the mean. We investigate expectations of power means of complex-valued random variables by using fractional calculus. We deal with both negative and positive orders of the fractional derivatives. We explicitly compute the expectations of the power means for both the univariate Cauchy distribution and the Poincaré distribution on the upper half-plane. We show that for these distributions the expectations are invariant with respect to the sample size and the value of the power. This talk will depend on [1].

REFERENCES

[1] K. Okamura, Y. Otobe, *Power means of random variables and characterizations of distributions via fractional calculus*, Probability and Mathematical Statistics 44 (1) (2024), 133–156.

OTROCOL, DIANA: *Fixed point results for non-self operators on $\mathbb{R}^m_+$-metric spaces.* (Joint work with Adela Novac and Veronica Ilea.)

The purpose of this paper is to study some problems of the fixed point theory for non-self operators on $\mathbb{R}^m_+$-metric spaces. The results complement and extend some known results given in the papers [1], [2], [3].

REFERENCES

[1] A. Chis-Novac, R. Precup, I.A. Rus, *Data dependence of fixed points for non-self generalized contractions*, Fixed Point Theory, 10 (1) (2009), 73–87.
[2] V. Ilea, D. Otrocol, I.A. Rus, M.A. Serban, *Applications of fibre contraction principle to some classes of functional integral equations*, Fixed Point Theory, 23 (1) (2022), 279–292.
[3] I.A. Rus, *Some variants of contraction principle, generalizations and applications*, Stud. Univ. Babeş-Bolyai Math., 61 (3) (2016), 343–358.

PASTECZKA, PAWEŁ: *Equality and comparison of generalized quasiarithmetic means.* (Joint work with Zsolt Páles.)

The purpose of this talk is to extend the definition of quasiarithmetic means by taking a strictly monotone generating function instead of a strictly monotone and continuous one. We establish the properties of such means and compare them to the analogous properties of standard quasiarithmetic means. The comparability and equality problems of generalized quasiarithmetic means are also solved. We also provide an example of a mean which, depending on the underlying interval or on the number of variables, could be or could not be represented as a generalized quasiarithmetic mean.

References

[1] Zs. Páles, P. Pasteczka, *Equality and comparison of generalized quasiarithmetic means*, preprint, *arXiv:2412.07315* (math.CA).

PINTÉR, MIKLÓS: *Functional equations in cooperative game theory.*

The talk aims to provide a concise introduction to the (potential) applications of functional equations and inequalities in the theory of transferable utility (TU) cooperative games [6]. The class of TU-games is a collection of finite-dimensional vector spaces, and the so-called solutions defined on this class (or a subclass) of TU-games. A solution is a set-valued mapping that assigns a "solution" – a subset of a finite-dimensional vector space – to each game. In particular, a value is a single-valued solution. The most well-known solutions include the Shapley value [4], the core [1, 5], and the nucleolus [3], among others.

To argue for or against a particular solution, the solution is characterized using so-called axioms. A characterization of a solution has the form: A FUNCTION defined on a specific SUBCLASS of TU-games is a SOLUTION if and only if it satisfies a given set of AXIOMS. In such cases, we say the AXIOMS characterize the SOLUTION on the SUBCLASS. These axioms take the form of functional equations and inequalities. In other words, a characterization result asserts that a set of functional equations and inequalities has a particular unique solution over a given domain.

For example, the Shapley value, which is a linear function, has numerous axiomatizations on various subclasses of TU-games. While (almost) each axiomatization is both interesting and significant, only three are considered "fundamentally" distinct: the one employing the linearity axiom [4], the one without the linearity axiom [7], and a recursive one [2].

References

[1] D.B. Gillies, *Solutions to general non-zero-sum games*, Contributions to the Theory of Games IV., Princeton University Press, 1959.

[2] G. Hamiache, *Associated consistency and Shapley value*, International Journal of Game Theory 30 (2001), 279–289.

[3] D. Schmeidler, *The Nucleolus of a Characteristic Function Game*, SIAM Journal on Applied Mathematics 17 (1969), 1163–1170.

[4] L.S. Shapley, *A value for n-person games*, In: H.W. Kuhn, A.W. Tucker (eds.), Contributions to the Theory of Games II, Annals of Mathematics Studies 28, Princeton University Press, Princeton, 1953, 307–317.

[5] L.S. Shapley, *Markets as Cooperative Games*, Tech. rep., Rand Corporation, 1955.

[6] J. von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton University Press, 1944.

[7] H.P. Young, *Monotonic Solutions of Cooperative Games*, International Journal of Game Theory 14 (1985), 65–72.

POPA, DORIAN: *On the best Ulam constant of a linear differential operator.* (Joint work with Alina-Ramona Baias.)

The linear differential operator with constant coefficients

$$D(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y, \quad y \in \mathcal{C}^n(\mathbb{R}, X)$$

acting in a Banach space X is Ulam stable if and only if its characteristic equation has no roots on the imaginary axis. We prove that if the characteristic equation of D has distinct roots r_k satisfying $\operatorname{Re} r_k > 0$, $1 \le k \le n$, then the best Ulam constant of D is

$$K_D = \frac{1}{|V|} \int_0^\infty \Big| \sum_{k=1}^{n} (-1)^k V_k e^{-r_k x} \Big| dx,$$

where $V = V(r_1, r_2, \ldots, r_n)$ and $V_k = V(r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_n)$, $1 \le k \le n$, are Vandermonde determinants.

RAŞA, IOAN: *Functional equations related to Appell polynomials and Heun functions (II).* (Joint work with Ana-Maria Acu.)

We are concerned with the functional equation

$$(x^2 - x)H_n'(x) = n(2x-1)\left(H_n(x) - H_{n-1}(x)\right), \ n \ge 1, \tag{1}$$

where $x \in \mathbb{R}$ and $H_0(x) = 1$.

It was investigated (see [1], [2], and the references therein) in relation with Legendre polynomials, Appell polynomials, Heun functions and certain entropies.

In this paper we consider functions of the form

$$H_n(x) = \sum_{k=0}^{n} b_{n,k}(x^2 - x)^k, \ n \ge 0,$$

where $x \in \mathbb{R}$, $b_{n,k} \in \mathbb{R}$, $b_{n,0} = 1$, $n \ge 0$.

We want to determine conditions on the coefficients $b_{n,k}$ under which $H_n(x)$ is

a) a solution to (1),
b) a Heun function,
c) a hypergeometric function.

Moreover, we study the functions $H_n(x)$ in connection with Appell polynomials, Gegenbauer (ultraspherical) polynomials and certain entropies.

REFERENCES

[1] A.M. Acu, I. Raşa, *Functional equations related to Appell polynomials and Heun functions*, Anal. Math. Phys. 12 (2022), 77.
[2] I. Raşa, *Convexity properties of some entropies (II)*, Results Math. 74 (2019), 154.

SABLIK, MACIEJ: *Social indices and bisymmetry.*

The Human Poverty Index is one of the measures used by the United Nations to compare countries of the world. It is admitted that roughly speaking the HPI is a power mean. However, there has been no consistency in choosing the power α. We show that the consistency may be obtained by solving the generalized bisymmetry equation. We also recall a result on representing the bisymmetrical operations in some function spaces.

SIKORSKA, JUSTYNA: *Is it still a characterization of the logarithmic mean?.*

Starting from the famous Hermite-Hadamard inequality for φ-convex functions and a characterization of the logarithmic mean we present some regularity results and outcomes on comparisons of means. Continuing the considerations from [1], we weaken the conditions and see whether the characterization remains valid.

REFERENCES

[1] T. Nadhomi, M. Sablik, J. Sikorska, *On a characterization of the logarithmic mean*, Results Math. 79 (2024), 200.

SZÉKELYHIDI, LÁSZLÓ: *Convolution-Type Functional Equations.*

In this talk we present some fundamental results in spectral synthesis – in terms of functional equations. We show that the basic problems – solved and unsolved – in spectral synthesis are closely related to the theory of convolution-type functional equations.

SZOSTOK, TOMASZ: *Orderings on measures induced by higher-order monotone functions.* (Joint work with Zsolt Páles.)

Let $I \subset \mathbb{R}$ be an interval. For a function $f\colon I \to \mathbb{R}$, for $n \in \mathbb{N} \cup \{0\}$ and for all pairwise distinct elements $x_0, \dots, x_n \in I$, we define

$$f[x_0, \dots, x_n] := \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{k=0,k\neq j}^{n}(x_j - x_k)},$$

which is called the *nth-order divided difference* for f at $x_0, \dots, x_n$. We say that f is *n-increasing* if, for all pairwise distinct elements $x_0, \dots, x_n \in I$, we have that $f[x_0, \dots, x_{n+1}] \geq 0$ (if the opposite inequality is true then we say that f is *n-decreasing*). Let μ be a non-zero bounded signed Borel measure on $[0, 1]$. We give a sufficient condition under which the inequality

$$(1) \qquad \int_{[0,1]} f\big((1-t)x + ty\big)d\mu(t) \geq 0, \qquad x, y \in I \text{ with } x < y,$$

is fulfilled for all functions f which are simultaneously k_1-increasing (or decreasing), k_2-increasing (or decreasing), $\dots$, k_l-increasing (or decreasing) for given nonnegative integers $k_1, \dots, k_l$. The necessary condition is also investigated. These results are applied to study the continuous solutions of linear functional inequalities of the form

$$\sum_{i=1}^{k} a_i f\big(\alpha_i x + (1-\alpha_i)y\big) \geq 0$$

where $\sum_{i=1}^{k} a_i = 0$ and of the inequality

$$\frac{1}{y-x}\int_x^y f(t)dt \leq \sum_{i=1}^{k} a_i f\big(\alpha_i x + (1-\alpha_i)y\big),$$

with $\sum_{i=1}^{k} a_i = 1$.

TÓTH, PÉTER: *Regularity preservation for quasisums.*

Let $n \geq 2$ be an integer and, for $k = 1, \dots, n$, let $f_k : I_k \longrightarrow \mathbb{R}$ be a continuous, strictly monotone function defined on a nonempty open interval. Moreover, suppose that the function $g : f_1(I_1) + \cdots + f_n(I_n) \longrightarrow \mathbb{R}$ is also continuous, strictly monotone. Then the mapping $F : I_1 \times \cdots \times I_n \longrightarrow \mathbb{R}$ defined by

$$F(x_1, \dots, x_n) = g\big(f_1(x_1) + \cdots + f_n(x_n)\big) \qquad (x_1 \in I_1, \dots, x_n \in I_n).$$

is called a quasisum.

In our talk we will show that if the quasisum F is continuously differentiable (in the Fréchet sense) then each of the generator functions $g, f_1, \dots, f_n$ is continuously differentiable as well. We will also present that the same holds for higher order continuous differentiability: if $p \in \mathbb{N}$ and F is p-times continuously differentiable, then $g, f_1, \dots, f_n$ are p-times continuously differentiable real functions.

2. Problems and Remarks

2.1. Problem.

Recently H. Alzer and J. Matkowski [1] have studied the following functional equation:

(1) $$f(x+y) = f(x)f(y) - \alpha xy, \quad x, y \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$ is a non-zero parameter and $f\colon \mathbb{R} \to \mathbb{R}$ is an unknown function. They proved two theorems on equation (1). The first result with a short proof [1, Theorem 1] completely describes solutions of (1) in case f has a zero. More precisely, they showed that if f solves (1) and it has a zero, then $\alpha > 0$ and either $f(x) = 1 - \sqrt{\alpha}x$, or $f(x) = 1 + \sqrt{\alpha}x$ for $x \in \mathbb{R}$. The second theorem with a longer proof [1, Theorem 2] provides the solutions to equation (1) under the assumption that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at least at one point. In this case, there are the same two solutions (clearly, both are differentiable and have a zero). In [1] the authors formulated the following conjecture:

CONJECTURE (ALZER AND MATKOWSKI). Every solution $f\colon \mathbb{R} \to \mathbb{R}$ of (1) has a zero.

This conjecture has been answered affirmatively by T. Małolepszy, see [4]. Togerher with Marta Pierzchałka and Gabriela Smejda [3] we have determined the solutions of a more general equation, namely

(2) $$f(x+y) = f(x)f(y) - \phi(x,y), \quad x, y \in X,$$

where X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\phi\colon X \times X \to \mathbb{K}$ is a biadditive functional and $f\colon X \to \mathbb{K}$ is a function. The motivation for such a generalization comes from an article by K. Baron and Z. Kominek [2], in which the authors, in connection with a problem proposed by S. Rolewicz [5], studied mappings defined on a real linear space with the additive Cauchy difference bounded from below by a bilinear functional. Our main result is the following theorem.

Theorem 1. Assume that ϕ and f solve (2) and

(3) $$\exists_{z_0 \in X}\, \phi(z_0, z_0) \neq 0.$$

Then there exists a unique constant $a \in \mathbb{K} \setminus \{0\}$ such that

$$f(x) = a\phi(x, z_0) + 1, \quad x \in X \tag{4}$$

and moreover

$$a^2\phi(x, z_0)^2 = \phi(x, x), \quad x \in X. \tag{5}$$

References

[1] H. Alzer and J. Matkowski, *Bilinearity of the Cauchy exponential difference*, Bull. Polish Acad. Sci. Math., online first, (2025).

[2] K. Baron, Z. Kominek, *On functionals with the Cauchy difference bounded by a homogeneous functional*, Bull. Polish Acad. Sci. Math., 51 (3) (2003), 301–307.

[3] W. Fechner, M. Pierzchałka, G. Smejda, *On a generalized conjecture by Alzer and Matkowski*, preprint: arXiv:2412.14756.

[4] T. Małolepszy, J. Matkowski, *Bilinearity of the Cauchy differences*, manuscript.

[5] S. Rolewicz, *Φ-convex functions defined on metric spaces*, J. Math. Sci. (N.Y.), 115 (5) (2003), 2631–2652.

W. Fechner

2.2. **Problem.**

Given a subinterval I of $\mathbb{R}$ and a strictly monotone continuous function $\phi\colon I \to \mathbb{R}$, denote by $\mathcal{M}_\phi$ the 2-variable quasi-arithmetic mean generated by φ. Further, for a number $p \in \mathbb{R}$, let M_p be the Hölder mean with exponent p. This is the quasi-arithmetic mean generated by the function $x \mapsto x^p$ $(x \in \mathbb{R}_+)$ if $p \neq 0$, and by the map $\ln$ in case $p = 0$. In [2], the separation of quasi-arithmetic means by Hölder ones was characterized in the following way.

Theorem 1 ([2], Theorem 4)**.** *Let I and J be proper subintervals of $\mathbb{R}_+$, and let $\phi\colon I \to \mathbb{R}$ and $\psi\colon J \to \mathbb{R}$ be strictly monotone continuous functions. Then in order that there exist a number $p \in \mathbb{R}$ such that $\mathcal{M}_\phi \leq M_p$ on I and $M_p \leq \mathcal{M}_\psi$ on J, it is necessary and sufficient that for all $x, y \in J$ and $t > 0$ with $tx, ty \in I$, the following inequality be valid: $\mathcal{M}_\phi(tx, ty) \leq t\mathcal{M}_\psi(x, y)$.*

One of the generalizations of quasi-arithmetic means are the Bajraktarević means which were introduced in the paper [1]. In the 2-variable symmetric case, they are defined as follows. Given a subinterval I of $\mathbb{R}$ and continuous functions $f, g\colon I \to \mathbb{R}$ such that $g > 0$ and f/g is strictly monotone, let

$$\mathcal{B}_{f,g}(x, y) = \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x) + f(y)}{g(x) + g(y)}\right) \qquad (x, y \in I).$$

In the special case where

$$I = \mathbb{R}_+,\ f(x) = x^p,\ g(x) = x^q \qquad (x \in \mathbb{R}_+)$$

for some numbers $p, q \in \mathbb{R}$, we obtain the Gini means defined by

$$\mathcal{G}_{p,q}(x, y) := \begin{cases} \left(\dfrac{x^p + y^p}{x^q + y^q}\right)^{\frac{1}{p-q}}, & \text{if } p \neq q \\ \exp\left(\dfrac{x^p \ln x + y^p \ln y}{x^p + y^p}\right), & \text{if } p = q \end{cases} \qquad (x, y \in \mathbb{R}_+).$$

Now we can formulate the following problem which is analogous to Theorem 1.

Problem 1. *Let I and J be proper subintervals of $\mathbb{R}_+$, and let $f_1, g_1\colon I \to \mathbb{R};\ f_2, g_2\colon J \to \mathbb{R}$ be continuous functions such that $g_1, g_2 > 0$ and f_1/g_1, f_2/g_2 are strictly monotone. Then is the following assertion valid? In order that there exist numbers $p, q \in \mathbb{R}$ such that $\mathcal{B}_{f_1,g_1} \le \mathcal{G}_{p,q}$ on I and $\mathcal{G}_{p,q} \le \mathcal{B}_{f_2,g_2}$ on J, it is necessary and sufficient that for all $x, y \in J$ and $t > 0$ with $tx, ty \in I$, the following inequality be valid: $\mathcal{B}_{f_1,g_1}(tx, ty) \le t\mathcal{B}_{f_2,g_2}(x, y)$.*

REFERENCES

[1] M. Bajraktarević, *Sur une équation fonctionnelle aux valeurs moyennes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske Ser. II, vol. 13 (1958), 243–248.

[2] Zs. Páles, *Nonconvex functions and separation by power means*, Math. Inequal. Appl., vol. 3 (2000), 169–176.

ZS. PÁLES

2.3. **Problem.**

Definition. Let $N \neq \emptyset$, $|N| < \infty$, and $v : \mathcal{P}(N) \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N and v are called the set of players, and a transferable utility cooperative game (henceforth game) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$.

Definition. $I^*(v) \doteq \{x \in \mathbb{R}^N\colon \sum_{i\in N} x_i = v(N)\}$ denotes the set of preimputations of a game $v \in \mathcal{G}^N$.

Definition. Given a game $v \in \mathcal{G}^N$, a coalition $S \in \mathcal{A}$ and a payoff vector $x \in \mathbb{R}^N$, then the *excess* of coalition S by the payoff vector x in the game v is $e(S, x) \doteq v(S) - x(S)$, where $x(S) \doteq \sum_{i\in S} x_i$.

Definition. The vector

$$E_v(x) := [\ldots \ge e_v(S, x) \ge \ldots : S \in \mathcal{A}^*]$$

is called the excess vector. It consists of all the excesses in non-increasing order. The lexicographical ordering between $x, y \in \mathbb{R}^n$ is the following: $x \le_L y$ if $x = y$ or if there exists k such that $x_k < y_k$ and for every $i < k$ it holds that $x_i = y_i$. The prenucleolus is the set of preimputations which lexicographically minimize the excess vectors over the set of preimputations, that is,

$$N^*(v) := \{x \in I^*(v)\colon E_v(x) \le_L E_v(y),\ \forall y \in I^*(v)\}\,.$$

The lexicographic center algorithm [1], [2], [3] is one of the most well-known algorithms for computing the (pre)nucleolus.

Consider a game $v \in \mathcal{G}^N$ and the following problem:

$$\begin{array}{rll} & t \to \min & \\ \text{s.t.} & e(S,x) \le t, & S \in \mathcal{P}^*(N) \\ & x \in I^*(v) & \\ & t \in \mathbb{R} & \end{array} \tag{1}$$

It is easy to see, that (1) has an optimal solution. Let the optimal value of (1) be denoted by t_1 and

$$X_1 = \{x \in I^*(v) \colon e(S,x) \leq t_1,\ \forall S \in \mathcal{P}^*(N)\}.$$

Let W_1 denote the fix set of (1), that is

$$W_1 = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R},\ \text{such that } e(S,x) = c_S,\ \forall x \in X_1\}.$$

For all $k \geq 2$ consider the following LP:

$$\begin{aligned} & t \to \min \\ \text{s.t.}\quad & e(S,x) \leq t, \quad S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r) \\ & x \in X_{k-1} \\ & t \in \mathbb{R} \end{aligned} \tag{2}$$

It is easy to see, that (2) has an optimal solution. Let the optimal value of (2) be denoted by t_k and

$$X_k = \{x \in X_{k-1} \colon e(S,x) \leq t_k,\ \forall S \in \mathcal{P}^*(N) \setminus (\cup_{r=1}^{k-1} W_r)\}.$$

Let W_k denote the fix set of (2), that is

$$W_k = \{S \in \mathcal{P}^*(N) \colon \exists c_S \in \mathbb{R},\ \text{such that } e(S,x) = c_S,\ \forall x \in X_k\}.$$

It is easy to see, that $t_k \geq t_{k+1}$, $X_k \supseteq X_{k+1}$ for all k and there exists a k^*, such that for all $l \geq k^*$ $X_l = X_{k^*}$.

[2, 3] proved that the lexicographic center algorithm returns with the prenucleolus. The above described algorithm is a modification of the lexicographic center algorithm by [1], although the modifications do not change the result of the algorithm. Therefore, we can say, that [2] and [3] proved the following theorem:

Theorem. $N^*(v) = X_{k^*}$.

Proposition. The prenucleolus on the class of games $\mathcal{G}^N$ meets the axioms: ETP, NP, COV.

Open problem. Give ◯ in the following claim: A function ψ on the class of games $\mathcal{G}^N$, $|N| > 2$, is the prenucleolus if and only if it meets the axioms ◯.

References

[1] G. Huberman, *The nucleolus and the essential coalitions*, In: A. Bensoussan, J. Lions (eds.) Analysis and Optimization of Systems, Proceedings of the Fourth International Conference, Versailles, Lecture Notes in Control and Information Sciences 28, Springer, 1980, 416–422.

[2] A. Kopelowitz, *Computation of the kernels of simple games and the nucleolus of n-person games*, rM-31, Mathematics Department, The Hebrew University of Jerusalem, 1967.

[3] M. Maschler, B. Peleg, L.S. Shapley, *Geometric properties of the kernel, nucleolus and related solution concepts*, Mathematics of Operations Research, 4 (4) (1979), 303–338.

[4] D. Schmeidler, *The Nucleolus of a Characteristic Function Game*, SIAM Journal on Applied Mathematics, 17 (1969), 1163–1170.

M. Pintér

3. LIST OF PARTICIPANTS

1. **ACU, Ana-Maria**, Lucian Blaga University of Sibiu, Sibiu, Romania,
   E-mail: anamaria.acu@ulbsibiu.ro.
2. **ALMIRA, José María**, Universidad de Murcia, Murcia, Spain,
   E-mail: jmalmira@um.es.
3. **BAIAS, Alina Ramona**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
   E-mail: baias.alina@math.utcluj.ro.
4. **BARON, Karol**, University of Silesia, Katowice, Poland,
   E-mail: karol.baron@us.edu.pl.
5. **BESSENYEI, Mihály**, University of Miskolc, Miskolc, Hungary,
   E-mail: mihaly.bessenyei@uni-miskolc.hu.
6. **BOROS, Zoltán**, University of Debrecen, Debrecen, Hungary,
   E-mail: zboros@science.unideb.hu.
7. **BURAI, Pál**, Budapest University of Technology and Economics, Budapest, Hungary,
   E-mail: buraip@math.bme.hu.
8. **CHMIELEWSKA, Katarzyna**, Kazimierz Wielki University, Bydgoszcz , Poland,
   E-mail: katarzyna.chmielewska@ukw.edu.pl.
9. **CHMIELIŃSKI, Jacek**, University of the National Education Comission, Kraków, Poland,
   E-mail: jacek.chmielinski@uken.krakow.pl.
10. **CHUDZIAK, Jacek**, University of Rzeszów, Rzeszów, Poland,
    E-mail: jchudziak@ur.edu.pl.
11. **FAZEKAS, Borbála**, University of Debrecen, Debrecen, Hungary,
    E-mail: borbala.fazekas@science.unideb.hu.
12. **FECHNER, Włodzimierz**, Lodz University of Technology, Łódź, Poland,
    E-mail: wlodzimierz.fechner@p.lodz.pl.
13. **FÖRG-ROB, Wolfgang**, University of Innsbruck, Innsbruck, Austria,
    E-mail: wolfgang.foerg-rob@uibk.ac.at.
14. **FORTI, Gian Luigi**, University of Milan, Milan, Italy,
    E-mail: gianluigi.forti@unimi.it.
15. **FRIPERTINGER, Harald**, University of Graz, Graz, Austria,
    E-mail: harald.fripertinger@uni-graz.at.
16. **GER, Roman**, Silesian University, Katowice, Poland,
    E-mail: romanger@us.edu.pl.
17. **GILÁNYI, Attila**, University of Debrecen, Debrecen, Hungary,
    E-mail: gilanyi@inf.unideb.hu.
18. **GŁAZOWSKA, Dorota**, University of Zielona Góra, Zielona Góra, Poland,
    E-mail: d.glazowska@im.uz.zgora.pl.
19. **GOPALAKRISHNA, Chaitanya**, Indian Statistical Institute, Bengaluru, India,
    E-mail: cberbalaje@gmail.com.
20. **GRÜNWALD, Richárd**, University of Debrecen, Debrecen, Hungary,
    E-mail: richard.grunwald@science.unideb.hu.
21. **GSELMANN, Eszter**, University of Debrecen, Debrecen, Hungary,
    E-mail: gselmann@science.unideb.hu.
22. **HORVÁTH, László**, University of Pannonia, Veszprém, Hungary,
    E-mail: horvath.laszlo@mik.uni-pannon.hu.

23. **IQBAL, Mehak**, University of Debrecen, Debrecen, Hungary,
E-mail: iqbal.mehak@science.unideb.hu.
24. **JABŁOŃSKI, Wojciech**, Jan Kochanowski University of Kielce, Kielce, Poland,
E-mail: wjablonski@ujk.edu.pl.
25. **KÉZI, Csaba**, University of Debrecen, Debrecen, Hungary,
E-mail: kezicsaba@eng.unideb.hu.
26. **KISS, Tibor**, University of Debrecen, Debrecen, Hungary,
E-mail: kiss.tibor@science.unideb.hu.
27. **KISS, Gergely**, Corvinus University of Budapest, Budapest, Hungary,
E-mail: kigergo57@gmail.com.
28. **LACZKOVICH, Miklós**, Eötvös Loránd University, Budapest, Hungary,
E-mail: miklos.laczkovich@gmail.com.
29. **LEWANDOWSKI, Michał**, Warsaw School of Economics, Warsaw, Poland,
E-mail: michal.lewandowski@sgh.waw.pl.
30. **LOSONCZI, László**, University of Debrecen, Debrecen, Hungary,
E-mail: losonczi08@gmail.com.
31. **MATKOWSKI, Janusz**, University of Zielona Góra, Zielona Góra, Poland,
E-mail: J.Matkowski@wmie.uz.zgora.pl.
32. **MENZER, Rayene**, University of Debrecen, Debrecen, Hungary,
E-mail: rayene.menzer@science.unideb.hu.
33. **MÉSZÁROS, Fruzsina**, University of Debrecen, Debrecen, Hungary,
E-mail: mefru@science.unideb.hu.
34. **MOLNÁR, Lajos**, University of Szeged, Szeged, Hungary,
E-mail: molnarl@math.u-szeged.hu.
35. **MOLNÁR, Gábor**, University of Nyíregyháza, Nyíregyháza, Hungary,
E-mail: molnar.gabor@nye.hu.
36. **NAGY, Gergő**, University of Debrecen, Debrecen, Hungary,
E-mail: nagyg@science.unideb.hu.
37. **NG, Che Tat**, University of Waterloo, Waterloo, Ontario, Canada,
E-mail: ctng@uwaterloo.ca.
38. **OKAMURA, Kazuki**, Shizuoka University, Shizuoka, Japan,
E-mail: okamura.kazuki@shizuoka.ac.jp.
39. **OTROCOL, Diana**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
E-mail: diana.otrocol@math.utcluj.ro.
40. **PÁLES, Zsolt**, University of Debrecen, Debrecen, Hungary,
E-mail: pales@science.unideb.hu.
41. **PASTECZKA, Paweł**, University of the National Education Comission, Kraków, Poland,
E-mail: pawel.pasteczka@up.krakow.pl.
42. **PINTÉR, Miklós**, Corvinus University of Debrecen, Budapest, Hungary,
E-mail: pmiklos@protonmail.com.
43. **POPA, Dorian**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
E-mail: popa.dorian@math.utcluj.ro.
44. **RAŞA, Ioan**, Technical University of Cluj-Napoca, Cluj-Napoca, Romania,
E-mail: ioan.rasa@math.utcluj.ro.
45. **SABLIK, Maciej**, University of Silesia, Katowice, Poland,
E-mail: maciej.sablik@us.edu.pl.

46. **SIKORSKA, Justyna**, University of Silesia, Katowice, Poland,
E-mail: justyna.sikorska@us.edu.pl.
47. **SOLARZ, Paweł**, University of the National Education Commission, Kraków, Poland,
E-mail: pawel.solarz@up.krakow.pl.
48. **SZÉKELYHIDI, László**, University of Debrecen, Debrecen, Hungary,
E-mail: lszekelyhidi@gmail.com.
49. **SZOKOL, Patrícia**, University of Debrecen, Debrecen, Hungary,
E-mail: szokol.patricia@inf.unideb.hu.
50. **SZOSTOK, Tomasz**, University of Silesia, Katowice, Poland,
E-mail: tomasz.szostok@us.edu.pl.
51. **TO, Lan Nhi**, University of Debrecen, Debrecen, Hungary,
E-mail: lan.nhi.to@science.unideb.hu.
52. **TÓTH, Péter**, University of Debrecen, Debrecen, Hungary,
E-mail: toth.peter@science.unideb.hu.